2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08

Lecture 18 - Solution of F.E. equations

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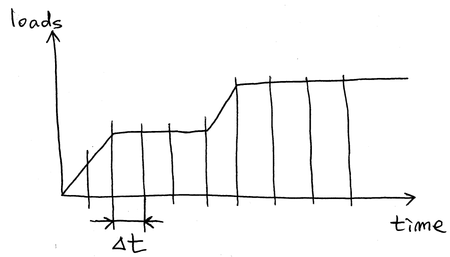
In structures,

Reading: Sec. 8.4

$$\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{p}) = \boldsymbol{R}. \tag{18.1}$$



In heat transfer,

$$\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{Q} \tag{18.2}$$

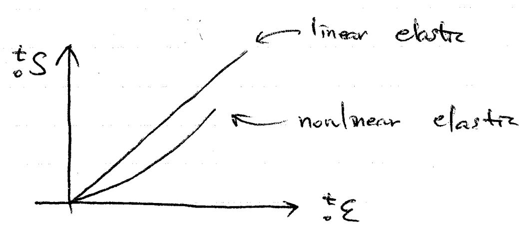
In fluid flow,

$$\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\theta}) = \boldsymbol{R} \tag{18.3}$$

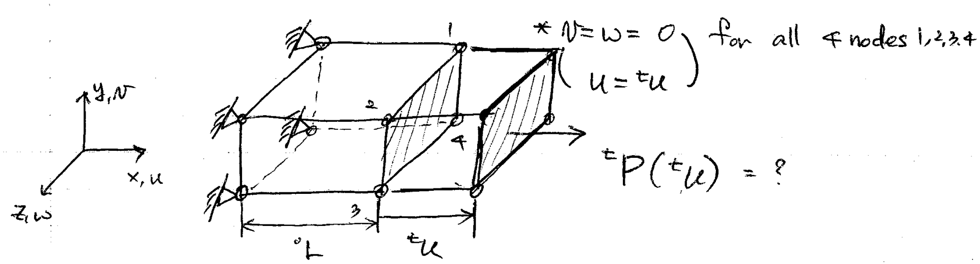
In structures/solids

$$\boldsymbol{F} = \sum_m \boldsymbol{F}^{(m)} = \sum_m \int_{^0V^{(m)}} {}^t_0\boldsymbol{B}_L^{(m)^T} \, {}^t_0\hat{\boldsymbol{S}}^{(m)} d^0V^{(m)} \tag{18.4}$$

Elastic materials



Example p. 590 textbook



Material law

$$ {}_0^tS_{11} = \tilde{E}\,{}_0^t\epsilon_{11} \tag{18.5}$$

In isotropic elasticity:

$$ \tilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \qquad (\nu = 0.3) \tag{18.6}$$

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left[\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right] \Rightarrow {}_0^t\epsilon_{11} = \frac{1}{2}\left[\left(\frac{{}^0L + {}^tu}{{}^0L}\right)^2 - 1\right] = \frac{1}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right] \tag{18.7}$$

where ${}_0^t\boldsymbol{U}$ is the stretch tensor.

$$ {}_0^tS_{11} = \frac{{}^0\rho}{{}^t\rho}\,{}_t^0X_{11}\,{}^t\tau_{11}\,{}_t^0X_{11}^T \tag{18.8}$$

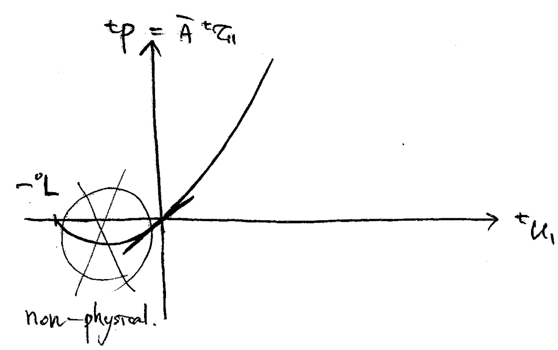
with

$$ {}_t^0X_{11} = \frac{{}^0L}{{}^0L + {}^tu}, \qquad {}^0\rho\,{}^0L = {}^t\rho\,{}^tL \tag{18.9}$$

$$ \Rightarrow {}_0^tS_{11} = \frac{{}^tL}{{}^0L}\left(\frac{{}^0L}{{}^tL}\right)^2 {}^t\tau_{11} = \frac{{}^0L}{{}^tL}\,{}^t\tau_{11} \tag{18.10}$$

$$ \therefore \frac{{}^0L}{{}^tL}\,{}^t\tau_{11} = \tilde{E}\cdot\frac{1}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right] \tag{18.11}$$

$$ \Rightarrow {}^t\tau_{11}\overline{A} = \boxed{{}^tP = \frac{\tilde{E}\overline{A}}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right]\left(1 + \frac{{}^tu}{{}^0L}\right)} \tag{18.12}$$



This is because of the material-law assumption (18.5) (okay for small strains . . .)

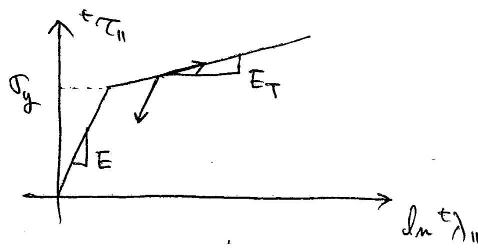
Hyperelasticity

$$ {}_0^tW = f(\text{Green-Lagrange strains, material constants}) \tag{18.13}$$

$$ {}_0^tS_{ij} = \frac{1}{2}\left(\frac{\partial\,{}_0^tW}{\partial\,{}_0^t\epsilon_{ij}} + \frac{\partial\,{}_0^tW}{\partial\,{}_0^t\epsilon_{ji}}\right) \tag{18.14}$$

$$ {}_0C_{ijrs} = \frac{1}{2}\left(\frac{\partial\,{}_0^tS_{ij}}{\partial\,{}_0^t\epsilon_{rs}} + \frac{\partial\,{}_0^tS_{ij}}{\partial\,{}_0^t\epsilon_{sr}}\right) \tag{18.15}$$

Plasticity



- yield criterion
- flow rule
- hardening rule

$$ {}^{t}\boldsymbol{\tau} = {}^{t-\Delta t}\boldsymbol{\tau} + \int_{t-\Delta t}^{t} d\boldsymbol{\tau} \tag{18.16} $$

Solution of (18.1) (similarly (18.2) and (18.3))

Newton-Raphson Find $\boldsymbol{U}^*$ as the zero of $f(\boldsymbol{U}^*)$

$$ \boldsymbol{f}(\boldsymbol{U}^*) = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F} \tag{18.17} $$

$$ = \boldsymbol{f}\left({}^{t+\Delta t}\boldsymbol{U}^{(i-1)}\right) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \cdot \left(\boldsymbol{U}^* - {}^{t+\Delta t}\boldsymbol{U}^{(i-1)}\right) + H.O.T. \tag{18.18} $$

where ${}^{t+\Delta t}\boldsymbol{U}^{(i-1)}$ is the value we just calculated and an approximation to $\boldsymbol{U}^*$.

Assume ${}^{t+\Delta t}\boldsymbol{R}$ is independent of the displacements.

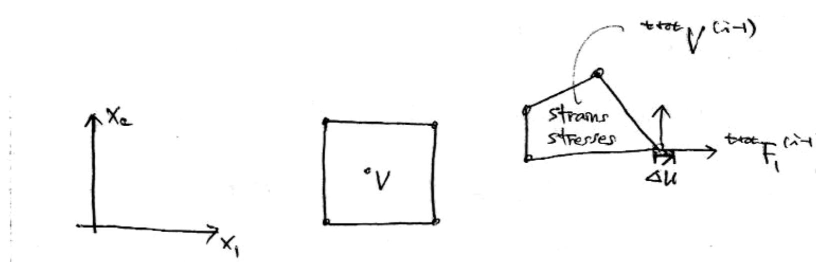
$$ \mathbf{0} = \left({}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\right) - \left.\frac{\partial\, {}^{t+\Delta t}\boldsymbol{F}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \cdot \Delta\boldsymbol{U}^{(i)} \tag{18.19} $$

We obtain

$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)} \Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{18.20} $$

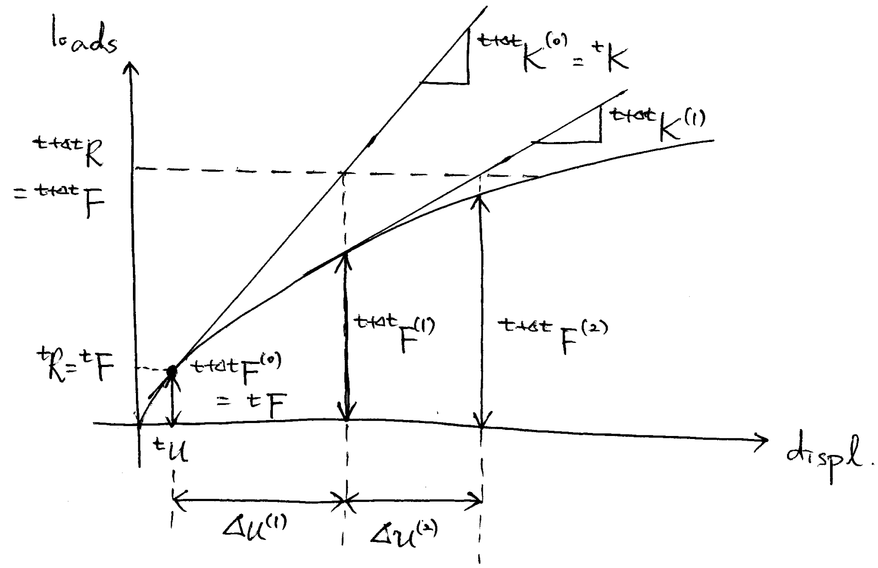
$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)} = \left.\frac{\partial\, {}^{t+\Delta t}\boldsymbol{F}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} = \left.\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}\right)\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \tag{18.21} $$

Physically



$$ {}^{t+\Delta t}K_{11}^{(i-1)} = \frac{\Delta\left({}^{t+\Delta t}F_1^{(i-1)}\right)}{\Delta u} \tag{18.22} $$

Pictorially for a single degree of freedom system



$$i = 1; \qquad {}^{t}K\Delta u^{(1)} = {}^{t+\Delta t}R - {}^{t}F \tag{18.23}$$

$$i = 2; \qquad {}^{t+\Delta t}K^{(1)}\Delta u^{(2)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(1)} \tag{18.24}$$

Convergence Use

$$\|\Delta \boldsymbol{U}^{(i)}\|_2 < \epsilon \tag{18.25}$$

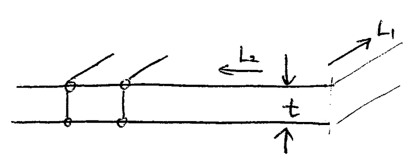
$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_i (a_i)^2} \tag{18.26}$$

But, if incremental displacements are small in every iteration, need to also use

$$\|{}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\|_2 < \epsilon_R \tag{18.27}$$

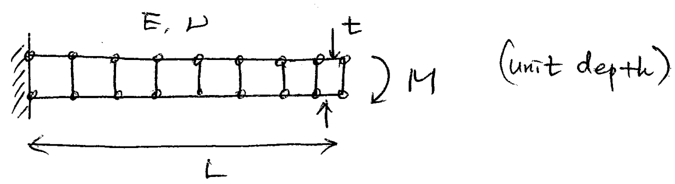
18.1 Slender structures

(beams, plates, shells)

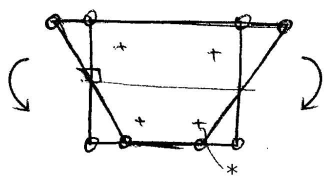


$$\frac{t}{L_i} \ll 1 \tag{18.28}$$

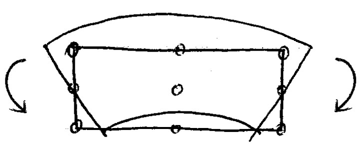
Beam



e.g. $\frac{t}{L} = \frac{1}{100}$



(4-node el.)
The element does not have curvature → we have a spurious shear strain



(9-node el.)
→ We do not have a shear (better)
→ But, still for thin structures, it has problems like ill-conditioning.

⇒ We need to use beam elements. For curved structures also spurious membrane strain can be present.