

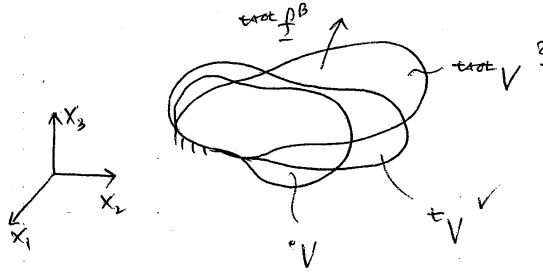
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2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

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## Lecture 14 - Total Lagrangian formulation, cont'd

Truss element. 2D and 3D solids.



$$\int_{t+\Delta t V} {}^{t+\Delta t} \tau_{ij} \delta {}^{t+\Delta t} e_{ij} d {}^{t+\Delta t} V = {}^{t+\Delta t} \mathcal{R} \quad (14.1)$$

$$\int_{0V} {}^{t+\Delta t} S_{ij} \delta {}^{t+\Delta t} \epsilon_{ij} \delta {}^0 V = {}^{t+\Delta t} \mathcal{R} \quad (14.2)$$

↓ linearization

$$\int_{0V} {}_0 C_{ijrs} \delta e_{rs} \delta e_{ij} \delta {}^0 V + \int_{0V} {}^t S_{ij} \delta \eta_{ij} \delta {}^0 V = {}^{t+\Delta t} \mathcal{R} - \int_{0V} {}^t S_{ij} \delta e_{ij} \delta {}^0 V \quad (14.3)$$

**Note:**

$$\delta {}_0 e_{ij} = \delta {}_0^t \epsilon_{ij}$$

varying with respect to the configuration at time  $t$ .

### F.E. discretization

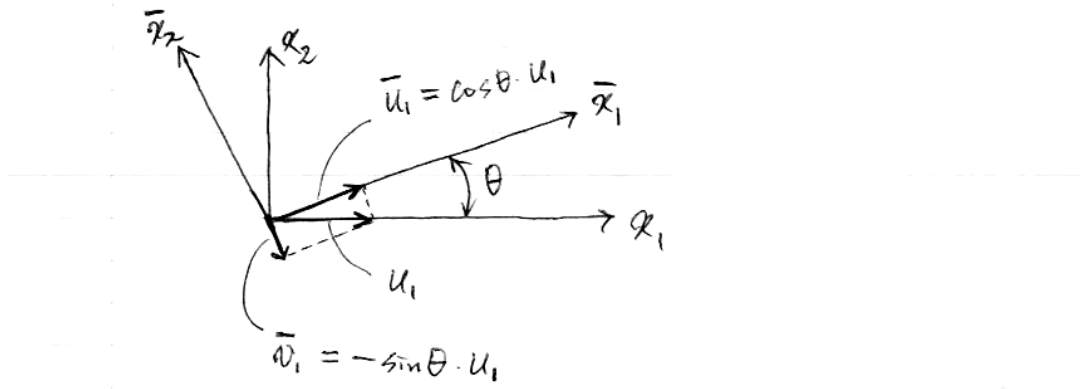
$${}^0 x_i = \sum_k h_k {}^0 x_i^k \quad {}^t x_i = \sum_k h_k {}^t x_i^k \quad {}^{t+\Delta t} x_i = \sum_k h_k {}^{t+\Delta t} x_i^k \quad (14.4a)$$

$${}^t u_i = \sum_k h_k {}^t u_i^k \quad {}^{t+\Delta t} u_i = \sum_k h_k {}^{t+\Delta t} u_i^k \quad u_i = \sum_k h_k u_i^k \quad (14.4b)$$

(14.4) into (14.3) gives

$$({}_0^t \mathbf{K}_L + {}_0^t \mathbf{K}_{NL}) \mathbf{U} = {}^{t+\Delta t} \mathbf{R} - {}^t \mathbf{F} \quad (14.5)$$

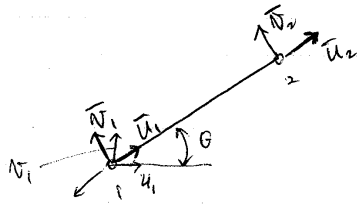
## Truss



$\frac{\Delta L}{L} \ll 1$  small strain assumption:

$$\begin{aligned}
 {}^t_0\mathbf{K} &= \frac{E^0 A}{L} \\
 &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta & -\sin \theta \cos \theta & -\sin^2 \theta \\ -\cos^2 \theta & -\cos \theta \sin \theta & \cos^2 \theta & \sin \theta \cos \theta \\ -\cos \theta \sin \theta & -\sin^2 \theta & \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \\
 &+ \frac{{}^tP}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
 \end{aligned} \tag{14.6}$$

(notice that the both matrices are symmetric)

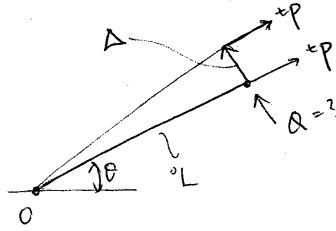


$$\begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \tag{14.7}$$

Corresponding to the  $\bar{u}$  and  $\bar{v}$  displacements we have:

$${}^t_0\mathbf{K} = \frac{E^0 A}{L} \tag{14.8}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{{}^tP}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{14.9}$$



$$Q^0 L = {}^t P \cdot \Delta \quad \Rightarrow \quad Q = \boxed{\frac{{}^t P}{{}^0 L}} \cdot \Delta \quad (14.10)$$

where the boxed term is the stiffness. In axial direction,  $\frac{{}^t P}{{}^0 L}$  is not very important because usually  $\frac{E^0 A}{{}^0 L} \gg \frac{{}^t P}{{}^0 L}$ . But, in vertical direction,  $\frac{{}^t P}{{}^0 L}$  is important.

$${}^0 \mathbf{F} = {}^t P \begin{bmatrix} -\cos \theta \\ -\sin \theta \\ \cos \theta \\ \sin \theta \end{bmatrix} \quad (14.11)$$

**2D/3D (e.g. Table 6.5) 2D:**

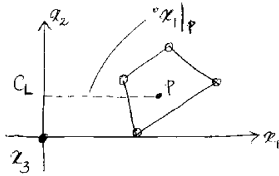
$${}^0 \epsilon_{11} = \underbrace{{}^0 u_{1,1} + {}^t u_{1,1} {}^0 u_{1,1} + {}^t u_{2,1} {}^0 u_{2,1}}_{{}^0 \epsilon_{11}} + \frac{1}{2} \underbrace{\left[ ({}^0 u_{1,1})^2 + ({}^0 u_{2,1})^2 \right]}_{{}^0 \eta_{11}} \quad (14.12)$$

$${}^0 \epsilon_{22} = \dots \quad (14.13)$$

$${}^0 \epsilon_{12} = \dots \quad (14.14)$$

(Axisymmetric)

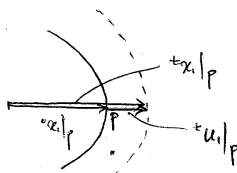
$${}^0 \epsilon_{33} = ? \quad (14.15)$$



$${}^t \epsilon = \frac{1}{2} \left( ({}^t \mathbf{U})^2 - \mathbf{I} \right) \quad (14.16)$$

$${}^0 \mathbf{U}^2 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{bmatrix} \quad (14.17)$$

↑  
( ${}^t \lambda$ )<sup>2</sup>



$$\begin{aligned} {}^t \lambda &= \frac{d^t s}{d^0 s} = \frac{2\pi ({}^0 x_1 + {}^t u_1)}{2\pi {}^0 x_1} \\ &= 1 + \frac{{}^t u_1}{{}^0 x_1} \end{aligned} \quad (14.18)$$

$$\begin{aligned} {}^t \epsilon_{33} &= \frac{1}{2} \left[ \left( 1 + \frac{{}^t u_1}{{}^0 x_1} \right)^2 - 1 \right] \\ &= \frac{{}^t u_1}{{}^0 x_1} + \frac{1}{2} \left( \frac{{}^t u_1}{{}^0 x_1} \right)^2 \end{aligned} \quad (14.19)$$

$${}^{t+\Delta t}{}_{0}\epsilon_{33} = \frac{{}^t u_1 + u_1}{{}_0 x_1} + \frac{1}{2} \left( \frac{{}^t u_1 + u_1}{{}_0 x_1} \right)^2 \quad (14.20)$$

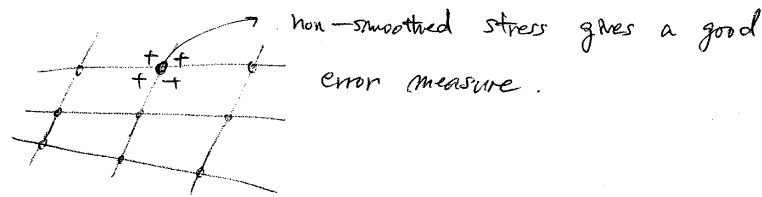
$${}_{0}\epsilon_{33} = {}^{t+\Delta t}{}_{0}\epsilon_{33} - {}^t\epsilon_{33} = \frac{u_1}{{}_0 x_1} + \frac{{}^t u_1}{{}_0 x_1} \cdot \frac{u_1}{{}_0 x_1} + \frac{1}{2} \left( \frac{u_1}{{}_0 x_1} \right)^2 \quad (14.21)$$

How do we assess the accuracy of an analysis?

Reading:  
Sec. 4.3.6

- Mathematical model  $\sim \mathbf{u}$
- F.E. solution  $\sim \mathbf{u}_h$

Find  $\|\mathbf{u} - \mathbf{u}_h\|$  and  $\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|$ .



## References

- [1] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - on Mesh Selection." *Computers & Structures*, 21:257–264, 1985.
- [2] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - Stress Band Plots and the Evaluation of Finite Element Meshes." *Journal of Engineering Computations*, 3:178–191, 1986.