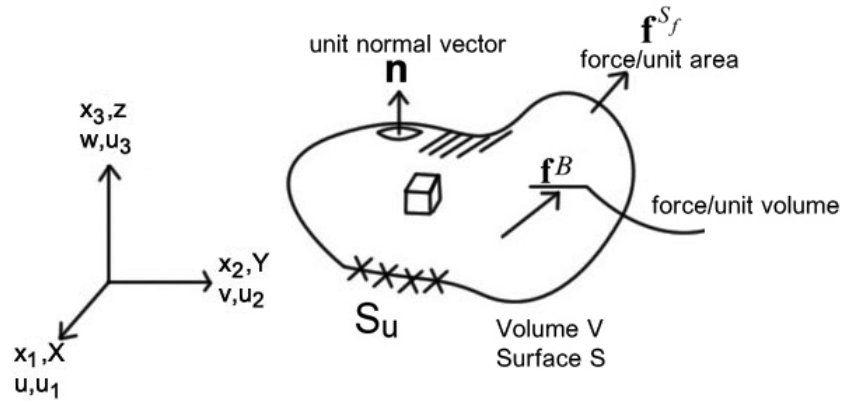


Lecture 4 - The Principle of Virtual Work

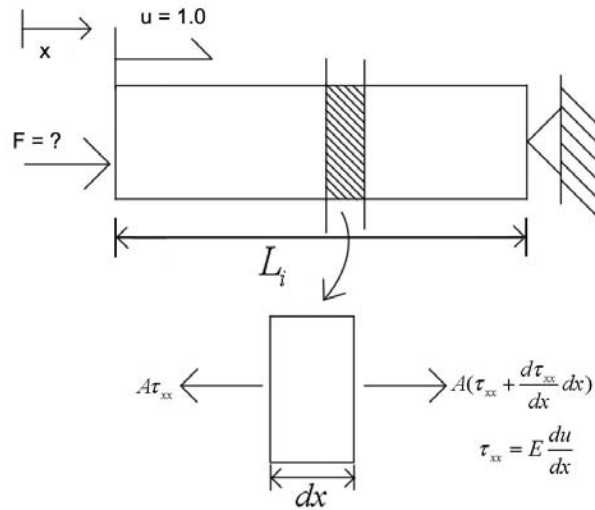


S_u = Surface on which displacements are prescribed
 S_f = Surface on which loads are applied
 $S_u \cup S_f = S$; $S_f \cap S_u = \emptyset$

Given the system geometry (V, S_u, S_f) , loads $(\mathbf{f}^B, \mathbf{f}^{S_f})$, and material laws, we calculate:

- Displacements u, v, w (or u_1, u_2, u_3)
- Strains, stresses

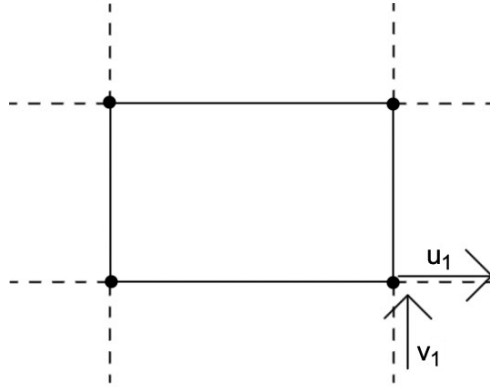
We will perform a linear elastic analysis for solids. We want to obtain the equation $\mathbf{KU} = \mathbf{R}$. Recall our truss example. There, we had element stiffness $\frac{AE}{L_i}$. To calculate the stiffnesses, we could proceed this way:



Every differential element should satisfy $EA \frac{d^2 u}{dx^2} = 0$. To obtain F, we solve:

$$EA \frac{d^2 u}{dx^2} = 0 \quad ; \quad u \Big|_{x=0} = 1.0 \quad ; \quad u \Big|_{x=L_i} = 0$$

Consider a 2D analysis:



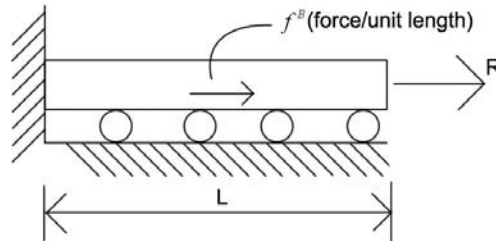
In this case, the method used for the truss problem to get the stiffness matrix \mathbf{K} would not work. In general 3D analysis, we must satisfy (for the *exact* solution)

- Equilibrium:
 - I. $\tau_{ij,j} + f_i^B = 0$ in $V(i, j = 1, 2, 3)$, where τ_{ij} are the Cauchy stresses (forces per unit area in the deformed geometry).
 - II. $\tau_{ij} n_j = f_i^{S_f}$ on S_f
- Compatibility: $u_i = u_i^{S_u}$ on S_u and all displacements must be continuous.
- Stress-strain laws

This is known as the differential formulation.

Example

Reading assignment: Section 3.3.4



- Equilibrium

$$EA \frac{d^2 u}{dx^2} + f^B = 0 \tag{a}$$

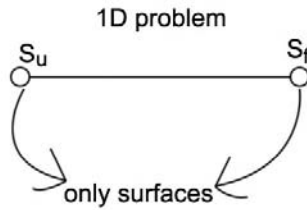
$$EA \frac{du}{dx} \Big|_{x=L} = R \tag{b}$$

- Compatibility

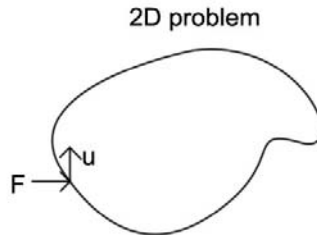
$$u \Big|_{x=0} = 0 \quad (c)$$

- Stress-strain law

$$\tau_{xx} = E \frac{du}{dx} \quad (d)$$



In a 1D problem, nodes are surfaces.



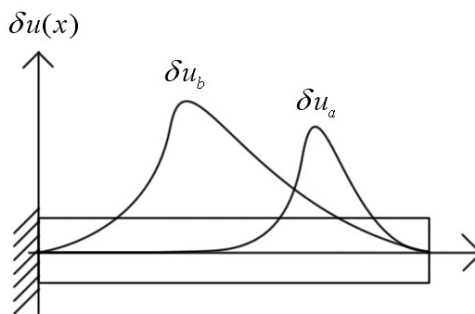
In a 2D problem, we define line \times thickness = surface, but one point can belong to both S_f and S_u .

Principle of Virtual Work (Virtual Displacements)

Clearly, the exact solution $u(x)$ must satisfy:

$$\left(EA \frac{d^2 u}{dx^2} + f^B \right) \delta u(x) = 0 \quad (1)$$

where $\delta u(x)$ is continuous and zero at $x = 0$. Otherwise, it is an arbitrary function. Hence, also,



$$\int_0^L \left(EA \frac{d^2 u}{dx^2} + f^B \right) \delta u(x) dx = 0 \quad (2)$$

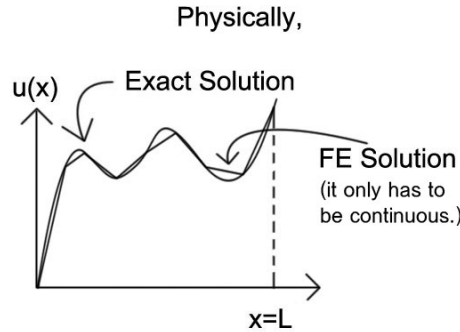
From Eq. (2):

$$EA \frac{du}{dx} \delta u \Big|_0^L - \int_0^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx + \int_0^L f^B \delta u dx = 0 \quad (\text{A})$$

The equation above becomes:

$$\underbrace{\int_0^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx}_{\text{Internal virtual work}} = \underbrace{\int_0^L f^B \delta u dx}_{\text{External virtual work}} + \underbrace{R \delta u \Big|_L}_{\text{Virtual work due to boundary forces}}$$

where $\frac{d\delta u}{dx}$ are the virtual strains, $\frac{du}{dx}$ are the real strains, and δu are the virtual displacements. We set $\delta u = 0$ on S_u , since we do not know the external forces on S_u . To solve $EA \frac{d^2 u}{dx^2} + f^B = 0$, we look for a function u where $\frac{d^2 u}{dx^2}$ exists ($\frac{du}{dx}$ should be continuous). In order to calculate the virtual work, we look for the solutions where only u is continuous.



(A) can be written as:

$$\int_0^L \bar{\epsilon}_{xx} EA \epsilon_{xx} dx = \int_0^L \bar{u} f^B dx + R \bar{u}_L \quad (\text{A}')$$

(the bar denotes 'virtual' quantities)

In 3D vector form, the principle of virtual work now becomes

$$\int_V \bar{\boldsymbol{\epsilon}}^T \mathbf{C} \boldsymbol{\epsilon} dV = \int_V \bar{\mathbf{u}}^T \mathbf{f}^B dV + \int_{S_f} \bar{\mathbf{u}}^{S_f T} \mathbf{f}^{S_f} dS_f \quad (\text{B})$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} ; \quad \epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \bar{\epsilon}_{xx} \\ \bar{\epsilon}_{yy} \\ \bar{\epsilon}_{zz} \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{bmatrix} ; \quad \bar{\epsilon}_{zz} = \frac{\partial \bar{u}}{\partial z}$$

We see that (B) is the generalized form of (A'). The principle of virtual work states that for any compatible virtual displacement field imposed on the body in its state of equilibrium, the total internal virtual work is

equal to the total external virtual work. Note that this variational formulation is equivalent to the differential formulation, given earlier.

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2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
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